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Self-similar flows with uniform velocity gradient and their use in modelling the free expansion of polytropic gases

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The self-similar motion of a polytropic gas with a linear velocity distribution is considered in an arbitrary ν-dimensional space. It is shown that if the initial state of the gas is isotropic the flow has a characteristic ellipsoidal form. Both expanding and compressing flows are shown to exist. The application of such flows as models for the expansion of an initially uniform mass of gas into vacuum is considered by comparison with computationally modelled expansions in one-dimensional cylindrical and spherical geometries. It is found that the accuracy of the representation increases when the heating time is long compared with the characteristic time of expansion.

1. Introduction

Self-similar solutions play an important role in gas dynamics by virtue of their relative mathematical simplicity. Sedov (1957) has given a general approach for the construction of such solutions in one dimension and shown how they may be applied to a variety of physical problems. These include the progressing wave solutions (Courant & Friedrichs 1948) as well as those considered here. In this paper we shall consider only the behaviour of those self-similar flows for which the velocity profile is linear in space. This work will present a generalization of Sedov's (1957) analysis to an arbitrary dimensional space.

The self-similar expansion into vacuum of a mass of gas with a linear velocity profile has proved to be a convenient model to study the development of such expansions analytically. The model has been widely explored in one-dimensional planar, cylindrical and spherical geometries (Sedov 1957; Keller 1956) for adiabatic flows of polytropic gases. The extension of the model to include isothermal flow appears in papers by Kulikovskii (1958), Korobeinikov & Ryazanov (1959), and Imshennik (1960). The model was further extended by Nemchinov (1961, 1964) to include heating of the gas in conformity with the self-similar expansion. This approach was further refined by Haught & Polk (1970) in a numerical study of the heating of small pellets by lasers.

The above studies were all performed in one dimension only. The generalization to three dimensions was made by Nemchinov (1965) and Dawson, Kaw & Green (1968) extending the model to include the expansion of a polytropic gas with an ellipsoïdal density distribution.

The application of this self-similar model is not limited to studies of expansion. Kidder (1974) has shown that by application of an appropriate pressure on the bounding surface, the model may also be used to describe the uniform adiabatic compression of spherical polytropic gas masses.

In this paper a general description of this type of self-similar flow in a space of arbitrary dimensions is developed: the above solutions appearing naturally within this description. It is found that in the practically important case when one of the initial-state variables is uniform, the flow is ellipsoidal in the Eulerian space of the flow, the eccentricities changing in time. Furthermore it is shown that such an ellipsoidal flow is self-similar.

The application of the self-similar, or similarity model is much restricted by the artificial nature of the initial distribution of state parameters introduced as a necessary consequence of the self-similar condition. However, as is well known, all expansions ultimately become self-similar in the asymptotic limit (Zel'dovich & Raizer 1966). Moreover as Nemchinov (1964) has pointed out and Fader (1968) demonstrated many flows rapidly approach a self-similar form. In consequence the similarity model may be used as an approximate description of such a flow, provided the appropriate matching parameters are known. This approach has been used by Haught & Polk (1970) and their followers in spherical geometry, and by the author (Pert 1976) for two dimensional cylindrical systems. Since the central assumption of this approach rests on the rather insecure foundation of insight (Nemchinov 1964) and a few indirect numerical tests (Fader 1968) we have examined the question in more detail. In this paper we present results of a numerical study of the expansions of one-dimensional cylinders and spheres to be compared with the equivalent self-similar model. In general we find that for the cases used practically the self-similar representation is remarkably good, and we present values of the appropriate matching parameters.

2. Self-similar expansion in a ν-dimensional space

We consider the dissipationless flow of a gas body of characteristic dimensions X_i which are functions of time only, in a ν-dimensional Cartesian geometry. We define a self-similar motion as one for which the co-ordinates of a given fluid element (x_i) remain in a fixed proportion to the characteristic dimensions (X_i) independent of time, i.e.

$$x_i/X_i = \xi_i, \tag{1}$$

where the ν values ξ_i are constant for a given fluid element. We may therefore regard ξ_i as a Lagrangian co-ordinate. The fluid velocity

$$u_i = \left.\frac{\partial x_i}{dt}\right|_{\xi_i} = \xi_i \frac{dX_i}{dt}.$$

Hence, if $U_i = dX_i/dt$,

$$u_i = U_i x_i/X_i. \tag{2}$$

This linear velocity dependence on the co-ordinate is also a sufficient condition, as well as a necessary one, for if equation (2) is valid then for a given particle

$$\frac{1}{x_i}\frac{dx_i}{dt} = \frac{1}{X_i}\frac{dX_i}{dt}$$

and equation (1) follows. Thus flows exhibiting either condition (1) or (2) may be regarded as self-similar for the present study.

The mass of a fluid cell of volume

$$d\tau = \prod_{i=1}^{\nu} dx_i,$$

is constant in time, as is the volume of the Lagrangian element

$$d\tau_\xi = \prod_{i=1}^{\nu} d\xi_i.$$

Thus, defining the Jacobian J

$$\frac{d\tau}{d\tau_\xi} = J = \frac{\partial(x_1, x_2, \ldots)}{\partial(\xi_1, \xi_2, \ldots)} = \prod_{i=1}^{\nu} X_i \tag{3}$$

we see that ρJ is an invariant of motion, and hence since J is a function of time alone, that the density ρ is a separable function of time t, and the Lagrangian space co-ordinates ξ_i.

Furthermore, it follows from equation (2) that the system must have reflexion symmetry in each co-ordinate plane, and therefore that ρ must be an even function of the Lagrangian variable ξ_i,

$$\rho = \rho_0(t) f(\xi_i^2) \tag{4}$$

where we have represented $f(\xi_1^2, \xi_2^2, \ldots)$ by $f(\xi_i^2)$, a form we shall use throughout.

The form of the function f is determined by Euler's equation which yields for the pressure p,

$$\frac{\partial p}{\partial \xi_i} = \xi_i f(\xi_i^2) \rho_0(t) X_i(t) \frac{dU_i}{dt} \tag{5}$$

and hence that the pressure is also a separable function of form

$$p = p_0(t)\,\phi(\xi_i^2), \tag{6}$$

where

$$p_0(t) = \lambda_i^{-1} \rho_0(t) X_i \frac{dU_i}{dt} \quad \text{for all } i, \tag{7}$$

and

$$\frac{\partial[\phi(\xi_i^2)]}{\partial(\xi_i^2)} = -\tfrac{1}{2}\lambda_i f(\xi_i^2) \tag{8}$$

where λ_i is a separation constant.

Since the pressure and density uniquely specify the thermodynamic state of the gas, all state variables must be separable functions of time and the Lagrangian variables ξ_i. Referring to equation (1) we observe that the co-ordinates x_i are themselves separable, and that an equivalent condition establishing equation (1) be that it be separable in time and the Lagrangian co-ordinates. We may therefore regard separability of this form as an equivalent condition for this self-similar motion, alternative to (1) or (2).

From equation (8) we observe that the initial condition of one state variable specifies the form of the spatial distribution f; and the temporal form of one (not necessarily the same) state variable the time development of the motion through (4) and (7) and the equation of state.

The equation of energy conservation in Lagrangian co-ordinates may be written:

$$\frac{d\epsilon}{dt} + p\frac{dV}{dt} = Q, \tag{9}$$

where ϵ is the specific internal energy, V the specific volume and Q the heat released per unit mass at that fluid element. If the motion is self-similar it is clear that the left-hand side of this equation is separable. It is therefore a necessary condition that the heat release function be of the form:

$$Q = Q_0(t)\, q(\xi_i^2). \tag{10}$$

In the particular case of a polytropic† gas of adiabatic index γ equation (9) takes the form:

$$q(\xi_i^2) = \mu\phi(\xi_i^2)/f(\xi_i^2) \tag{11}$$

and

$$Q_0(t) = \mu^{-1}\left\{\frac{1}{(\gamma-1)}\rho_0(t)\,\dot{p}_0(t) - \frac{\gamma}{(\gamma-1)}p_0(t)\,\dot{\rho}_0(t)/[\rho_0(t)]^2\right\} \tag{12}$$

where μ is a separation constant.

The heat release term may be generalized to include both internal (chemical reactions, nuclear energy release) and external (laser heating) sources, and energy redistribution (thermal conduction). In most cases the heat release function $q(\xi_i^2)$ will be specified, which in turn defines the self-similar distribution functions $f(\xi_i^2)$ and $\phi(\xi_i^2)$ through equations (8) and (11). For example in the case of a uniform internal source, $q(\xi_i^2)$ is a constant,

$$q(\xi_i^2) = \phi(\xi_i^2)/f(\xi_i^2) = \text{const.} \tag{13}$$

The gas temperature is therefore constant and the flow isothermal (Nemchinov 1965).

In principle, equation (12) allows a determination of the temporal development of the flow, however, in many cases the equation of overall energy conservation to be derived subsequently from equation (28) provides a simpler approach.

Ellipsoidal flows

Of special importance are isotropic systems where the initially specified state variable has no preferred direction in the Lagrangian space, ξ_i, i.e. it is a function of the Lagrangian 'radius' variable,

$$\zeta^2 = \sum_{i=1}^{\nu} \xi_i^2 = \sum_{i=1}^{\nu} (x_i/X_i)^2 \tag{14}$$

which in Cartesian space is an ellipsoidal variable. Since the equation of state implies:

$$\phi(\xi_i^2) = S[\zeta^2, f(\xi_i^2)] \tag{15}$$

equation (8) requires that ϕ and f be functions of ζ^2 only, and hence that all state variables be functions of ζ^2 alone, i.e. ellipsoidal distributions. The most usual class of initial conditions is isometric, i.e. a uniform distribution of some specified state variable, e.g. temperature (isothermal) entropy (isentropic) or density (isostatic). No isobaric (constant pressure) solutions exist.

We define an ellipsoidal flow as one for which the state variables are always functions of the ellipsoidal variable ζ, and time t, alone. Such flow is peculiar to self-similar motion of the type considered here. Thus the equation of continuity takes the form

$$\frac{\partial\rho}{\partial t} + \sum_{i=1}^{\nu}\left\{\frac{u_i\xi_i}{X_i} - \frac{U_i\xi_i^2}{X_i}\right\}\frac{1}{\zeta}\frac{\partial\rho}{\partial\zeta} + \rho\frac{\partial u_i}{\partial x_i} = 0 \tag{16}$$

† A polytropic gas is one whose internal energy is directly proportional to its temperature (Courant & Friedrichs 1948).

from which we conclude that the velocity components u_i, take the form:

$$u_i = \xi_i U_i g(\zeta, t) \tag{17}$$

and equation (16) takes the form

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{\nu} \left\{ \frac{U_i \xi_i^2}{X_i} \frac{1}{\zeta} \frac{\partial}{\partial \zeta} [\rho(g-1)] \right\} + \sum_{i=1}^{\nu} \frac{U_i}{X_i} \rho g = 0. \tag{18}$$

If this equation is independent of ξ_i, either

$$\frac{U_1}{X_1} = \frac{U_2}{X_2} = \dots \quad \text{or} \quad \rho(g-1) = h_1(t). \tag{19}$$

Euler's equation takes the form

$$U_i \frac{\partial q}{\partial t} + g \frac{dU_i}{dt} + \frac{U_i^2}{X_i} g(g-1) + U_i \sum_{j=1}^{\nu} \frac{U_j \xi_j^2}{X_j} (g-1) \frac{1}{\zeta} \frac{\partial g}{\partial \zeta} = -\frac{1}{X_i} \frac{1}{\rho \zeta} \frac{\partial p}{\partial \zeta} \tag{20}$$

and is only independent of ξ_i if

$$\frac{U_1}{X_1} = \frac{U_2}{X_2} = \dots \quad \text{or} \quad (g-1)^2 = h_2(t). \tag{21}$$

It is clear from the nature of the solution that $U_1/X_1 = U_2/X_2 = \dots$ if and only if $X_1 = X_2 = \dots$, i.e. the motion is spherical. The other conditions require either $g = 1$ or that ρ and g be functions of time alone, and therefore that the velocity distribution is given by (2). We therefore conclude that ellipsoidal flows are necessarily self-similar.

This result may be further generalized to cover flows in which an initially ellipsoidal distribution moves in such a way that a family of similar ellipsoids fixed in the fluid remain similar ellipsoids throughout the flow (although of course, the eccentricities change in time). It is evident that the flow in ξ_i space is radial. In consequence, the distribution between two neighbouring ellipsoids will remain uniform, i.e., the distribution is ellipsoidal at all times. Hence by the above theorem the motion is self-similar at all times.

Energy conservation

The total energy at any time in the gas $E(t)$ is the sum of the kinetic (E_k) and thermal (E_t) energies:

$$\begin{aligned} E_k &= \int \rho \tfrac{1}{2} \sum_{i=1}^{\nu} u_i^2 \, d\tau \\ &= \tfrac{1}{2}[J\rho_0(t)] \int f(\xi_i^2) \sum_{i=1}^{\nu} \xi_i^2 U_i^2 \, d\tau_\xi \end{aligned} \tag{22}$$

and

$$E_t = \int \rho C_v T \, d\tau = \frac{1}{(\gamma-1)} \int p \, d\tau \tag{23}$$

for an ideal gas of specific heat C_v and temperature T. For an expansion into vacuum, the pressure is zero on the boundary of the gas:

$$\begin{aligned} \int p \, d\tau &= -\int \sum_{i=1}^{\nu} x_i \frac{\partial p}{\partial x_i} d\tau = \int \sum_{i=1}^{\nu} \rho x_i \frac{du_i}{dt} d\tau \\ &= [J\rho_0(t)] \int f(\xi_i) \sum_{i=1}^{\nu} \xi_i^2 X_i \frac{dU_i}{dt} d\tau_\xi. \end{aligned} \tag{24}$$

These equations can be markedly simplified in the case of ellipsoidal flow, where f is a function of ζ^2 only:

$$\int \xi_i^2 f(\xi_i^2)\, d\tau_\xi = 1/\nu \int \zeta^2 f(\zeta^2)\, d\tau_\xi \tag{25}$$

by symmetry. Hence defining

$$\Upsilon = \tfrac{1}{2} \int \zeta^2 f(\zeta^2)\, d\tau_\xi / \int f(\zeta^2)\, d\tau_\xi \tag{26}$$

and the total mass

$$M = \int \rho\, d\tau = [J\rho_0(t)] \int f(\zeta^2)\, d\tau_\xi, \tag{27}$$

we obtain

$$\sum_{i=1}^{\nu} \left\{ U_1^2 + \gamma^1 X_i \frac{dU_i}{dt} \right\} = \nu E(t)/\Upsilon M, \tag{28}$$

where $\gamma^1 = 2/(\gamma-1)$. This equation, together with equation (7) in the form:

$$X_1 \frac{dU_1}{dt} = X_2 \frac{dU_2}{dt} = \dots \tag{29}$$

uniquely determine the expansion. These equations are of a form which is convenient for numerical integration.

We may note that motions with differing initial gas states, i.e. different forms of f and values of E and M, are similar, being parametrized by the aspect ratios (or eccentricities) of the original ellipsoid, the constant, γ, and the dimensionless form of the energy $e(t) = E(t)/E_0$.

Adiabatic flows

If the entropy everywhere in the gas remains constant in time, although it may vary spatially, the flow is adiabatic. The spatial pressure and density distributions must satisfy the adiabatic equation of state

$$\phi(\xi_i^2) = f^{\gamma}(\xi_i^2)\, g(\xi_i^2), \tag{30}$$

where g is a function expressing the initial spatial entropy distribution in the gas. Hence from (8)

$$\gamma g f^{(\gamma-2)}\, \partial f/\partial(\xi_i^2) + f^{(\gamma-1)}\, \partial g/\partial(\xi_i^2) = -\lambda_i, \tag{31}$$

where λ_i is a separation constant which determines the relationship of the variables X_i to the boundary. In the case of ellipsoidal flow, the distribution takes the one-dimensional form, whose isometric forms are well known (Sedov 1957; Keller 1956; Zel'dovich & Raizer 1966). For example isentropic forms are:

$$\begin{aligned} f^{(\gamma-1)} &= f_0^{(\gamma-1)}[1-(\zeta/\zeta_0)^2], \\ \phi^{(\gamma-1)/\gamma} &= \phi_0^{(\gamma-1)/\gamma}[1-(\zeta/\zeta_0)^2]. \end{aligned} \tag{32}$$

Isothermal flows

If the gas is everywhere at the same temperature, which may vary in time, the profile is Gaussian (Imshennik 1960; Fader 1968; Pert 1974)

$$f = f_0 \exp[-(\zeta/\zeta_0)^2]. \tag{33}$$

We may note that in the absence of heat input, isothermal flow is also adiabatic, a result of some practical significance (Pert 1974).

Self-similar compression

Although we have implicitly assumed in the foregoing discussion that the gas is expanding, this need not be the case, and the equations derived also represent uniform compression of the medium. In this case the boundary condition on the surface of the gas is somewhat different, namely a finite pressure. In addition since the flow is inward the separation constants, λ_i, are negative. Thus for an ellipsoidal adiabatic isentropic compression the density and pressure profiles take the form

$$\left.\begin{aligned} f^{(\gamma-1)} &= f_0^{(\gamma-1)}[1+(\zeta/\zeta_0)^2], \\ \phi^{(\gamma-1)/\gamma} &= \phi_0^{(\gamma-1)/\gamma}[1+(\zeta/\zeta_0)^2], \end{aligned}\right\} \tag{34}$$

in contrast to equation (32) for expansion. The equation of motion for adiabatic flow has the form

$$\frac{X_1\dfrac{dU_1}{dt}}{\left(X_1\dfrac{dU_1}{dt}\right)_0} = \frac{X_2\dfrac{dU_2}{dt}}{\left(X_2\dfrac{dU_2}{dt}\right)_0} = \dots = \left(\frac{J_0}{J}\right)^{\gamma-1}. \tag{35}$$

In the case of $\gamma = \frac{5}{3}$ the complete integral may be evaluated for spherical motion (Kidder 1974).

3. The similarity solution as an asymptotic form of a general expansion

In general we are not interested in the expansion into vacuum of a mass of gas with an artificial mass distribution such as for example (32) or (33) but rather ones with an arbitrary initial form. The importance of self-similar solutions lies in the fact that they represent the asymptotic motion of the arbitrary mass for large times. This may be demonstrated by the following generalization of an argument given by Zel'dovich & Raizer (1966).

Consider the expansion into vacuum from the rest of a three-dimensional mass of gas M, with characteristic dimensions X, Y and Z and initial thermal energy E. The gas pressure at any time $p \sim E_{th}/XYZ$, where E_{th} is the total thermal energy at time t, which is necessarily less than E due to the kinetic energy. The pressure gradients are of order p/X, p/Y and p/Z in the x, y and z directions respectively, and the density $\sim M/XYZ$. Hence the Lagrangian equation of motion for the characteristic dimension X is

$$\frac{dU}{dt} \approx \frac{E_{th}}{MX}, \quad U = \frac{dX}{dt}. \tag{36}$$

As time increases, the thermal energy decreases owing to the increase in kinetic energy. Since the thermal energy must be bounded, we may suppose that in general it can be represented by a form $E_{th} \approx E_0 + E_1 g(X) + E_2 h(X)$, where $g(X) \to 0$ faster than $X^{-\delta}$, $\delta > 0$ as $X \to \infty$, and $h(X)$ is an oscillatory function of constant or decreasing amplitude as X increases. Therefore

$$\tfrac{1}{2}U^2 = \tfrac{1}{2}U_0^2 + E_0/M \ln(X) + E_1/Mg'(X) + E_2/Mh'(X), \tag{37}$$

where $g'(X) \to 0$ faster than $X^{-\delta}$, as $X \to \infty$, and $h'(X)$ is an oscillatory function of decreasing amplitude. Since the kinetic energy of the fluid element associated with the characteristic dimension X is bounded we may conclude that $E_0 = 0$. Therefore since E_{th} is bounded from below (by 0) we conclude that $E_{th} \to 0$ faster than $X^{-\delta}$, as $X \to \infty$.

Applying this result to the Lagrangian equation of motion in the x direction for an arbitrary fluid element, we find that

$$u \to \text{const.} \quad \text{as} \quad X \to \infty.$$

Hence by a similar argument for the y and z directions we find that the respective velocity components v and w are also constant. Therefore in the limit $t \to \infty$:

$$x \to ut; \quad y \to vt; \quad z \to wt. \tag{38}$$

The velocity distribution therefore becomes self-similar in the asymptotic limit. Hence the state variables are described by separable functions in this limit.

Unfortunately we cannot determine the spatial distribution functions, f, ϕ, etc. for the flow in this limit, without following the flow from its inception. It might be expected, for example, in an isentropic flow, since the fluid always remains isentropic that the asymptotic density profile would approach the corresponding self-similar isentropic profile, equation (32). However, as the limit is only reached as $p \to 0$ and $\rho \to 0$, when the velocities are constant, the actual profile, which is maintained, is that which is developed prior to the asymptotic region of the flow (Zel'dovich & Raizer 1966). Nonetheless we may expect, on physical grounds, that the actual flow profile will closely resemble the self-similar one in the asymptotic limit, and that the flow will develop this form in a characteristic time $\sim L/C_0$, where L is a length characteristic of the initial dimensions, and C_0 the initial sound speed. This conclusion is borne out by detailed studies of the flow in special cases (Fader 1968).

There is one further case of flows in which a self-similar profile is established, namely the case of free molecular flow (Keller 1948; Molmud 1960). If the gas initially has a Maxwellian velocity distribution, the asymptotic density profile is Gaussian, as for isothermal flows.

4. The approximation of real expansions by equivalent self-similar flows

The simplicity of the form of the differential equation (28) allows a complete solution to the problem of expanding gas flow to be made with a minimum of numerical effort.

Unfortunately these solutions are only available for restricted spatial density profiles, which depend on the physical constraints of the problem. In particular one is usually concerned with the expansion into vacuum of an initially uniform gas mass of restricted physical dimensions (hereafter called the 'test distribution'), e.g. a slab, a cylinder or a sphere, whose initial density profile has the form:

$$\rho = \begin{cases} \rho(0) & \text{for} \quad x_i < X_i, \\ 0 & \text{for} \quad x_i > X_i. \end{cases} \tag{39}$$

The time development of such a gas mass involves a complete spatial and temporal solution of the equations of hydrodynamics, and is correspondingly considerably more difficult. However, one may argue on physical grounds that the final self-similar profile obtained must have a form which approximates to the equivalent self-similar form, e.g. an isothermal expansion will ultimately take on a Gaussian form.

If this hypothesis is correct, we may use the self-similar model to approximate the behaviour of an expanding gas mass in an averaged sense, with a considerable reduction in the computational effort required. Thus, for example, in many problems in the expansion of laser produced plasma one is interested in the gross parameters of the expanding gas mass rather than in its detailed structure. In such cases the self-similar form may be used to give a reasonable approximation of the complete history of the expansion of a uniform gas mass, the averaged values of the various variables being equivalanced.

In order to carry out this simulation we must define an equivalent self-similar ellipsoid to the uniform gas mass. It is clearly sensible to identify the form of the profile with that of the corresponding self-similar flow (isothermal, isentropic etc.), of the same total mass. The initial dimensions of the uniform gas mass x_0, y_0, z_0 must be expressed in terms of the initial parameters X_0, Y_0, Z_0 of the ellipsoid by

$$X_0 = \Gamma_x x_0, \quad Y_0 = \Gamma_y y_0, \quad Z_0 = \Gamma_z z_0. \tag{40}$$

It follows from a dimensional argument that the factors Γ_i depend on the aspect ratios,

$$l_1 = x_0/y_0, \quad l_2 = x_0/z_0, \tag{41}$$

only, for a constant total-energy expansion. If the fluid is heated during the expansion the additional parameters E_∞/E_0 and $L\tau/C_0$, where τ is the duration of the heating pulse, must be considered. Indeed when the heating pulse is long $(L\tau/C_0 \gg 1)$ Nemchinov (1964) has pointed out that we may expect that the body of gas will possess a self-similar form throughout much of its motion.

In assigning values to the real body of the gas based on the equivalent self-similar expansion we must ensure that the values so obtained remain physical. In particular, the averaged density of the self-similar profile may, for early times, when the test gas profile is still nearly square, exceed the initial density of the test gas distribution. This is clearly unphysical, but is easily circumvented by the use of a simple switch function of the form:

$$\bar{\rho} = \rho(0) \quad \text{if} \quad \bar{\rho}_s > \rho(0), \quad \text{otherwise} \quad \bar{\rho} = \bar{\rho}_s, \tag{42}$$

where $\bar{\rho}_s$ is the mean density calculated from the similarity expansion. In cases of three-dimensional expansion where the aspect ratios are finite and not unity, it may be necessary to generalize equation (42) to allow phases of dominant one- or two-dimensional expansion to take place initially due to the different characteristic times for establishing a similarity form in the different directions; this point is discussed in more detail in connexion with isothermal expansions in Pert's 1979 paper.

Alternatively we may consider the expansion of the test distribution in terms of its self-similar asymptotic form. From equation (28) we can observe that this may be described by means of the characteristic term Ψ. This approach, whilst automatically yielding the correct solution in the asymptotic limit, has the disadvantage that an *a priori* knowledge of the asymptotic distribution is required to evaluate Ψ. In the subsequent sections the calculations of the distributions are given for a variety of cases and the values of Ψ tabulated (with the factors $\Gamma = 1$, i.e. with the similarity variables X_i defined by the edge of the distribution). In order to make practical use of the solutions thus generated we define the factors

$$\begin{aligned}\alpha = \rho_0/\langle\rho\rangle &= \left(\int_0^1 \zeta^{\nu-1}\,d\zeta\right)\bigg/\left(\int_0^1 \zeta^{\nu-1}f(\zeta)\,d\zeta\right),\\ \beta = \rho_0/\bar{\rho} &= \left(\int_0^1 \zeta^{\nu-1}f(\zeta)\,d\zeta\right)\bigg/\left(\int_0^1 \zeta^{\nu-1}[f(\zeta)]^2\,d\zeta\right),\end{aligned} \tag{43}$$

where $\langle\rho\rangle$ is the average density defined by volume, $\bar{\rho}$ is the mean density defined by mass given subsequently in equation (44) and ρ_0 is the density at the centre.

We shall, in the remainder of this paper, examine the validity of this assumption by comparing computer calculations of the development of the test distribution with those of the similarity model. In this paper we shall restrict ourselves to considering one dimensional flow in cylindrical and spherical geometry. The extension to cases of two and three dimensions, whilst straightforward in principle offers problems in detail since the final distribution is not necessarily a function of the variable ζ^2. Thus a simple representation by a self-similar distribution which is a function of ζ^2 alone may not be possible.

5. One-dimensional expansions

In this section we compare the expansion into vacuum of cylindrical and spherical uniform gas masses with the equivalent similarity solution. The flow of the uniform gas mass is evaluated numerically using a standard one-dimensional Lagrangian code (Richtmyer & Morton 1967) with 100 cells. Since the expansion may be treated in terms of dimensionless quantities, we may specify the initial conditions quite generally by means of arbitrary values. For this purpose we have used an initial radius $R_0 = 1$, specific energy $E_0/M = 1$ and density $\rho(0) = 1$.

In order to compare the motion of the uniform gas mass with the equivalent similarity solution we define some simple comparison quantities. These are a mean density

$$\bar{\rho} = \int \rho\,dm\Big/\int dm = \int \rho^2\,d\tau\Big/\int \rho\,d\tau, \tag{44}$$

where dm is an element of mass, which broadly describes the central dense region of gas, and a parameter R_e related to the periphery of the expansion.

Adiabatic flows

Figure 1 shows the computed density profiles at time 10000 for both spherical (a) and cylindrical (b) geometries. At this time the profiles represent a close approximation to the final asymptotic profiles. In order to show the approach to the similarity form we have plotted the relative density $\rho(r,t)/\rho(0,t)$ against the parameter $1-(r/r_0)^2$ where r_0 is the edge of the outermost cell of the fluid. For comparison the similarity profile is plotted at the same time. Some marked differences are immediately apparent. The central density is always larger for the similarity model than for the real flow reflecting an overall faster expansion of the initially uniform gas. In the case of spherical expansion the asymptotic velocities of the plasma edge show close agreement for model and real systems. For cylindrical expansions the agreement is much poorer. From table 1 we observe that in contrast the central density is better modelled for cylindrical systems.

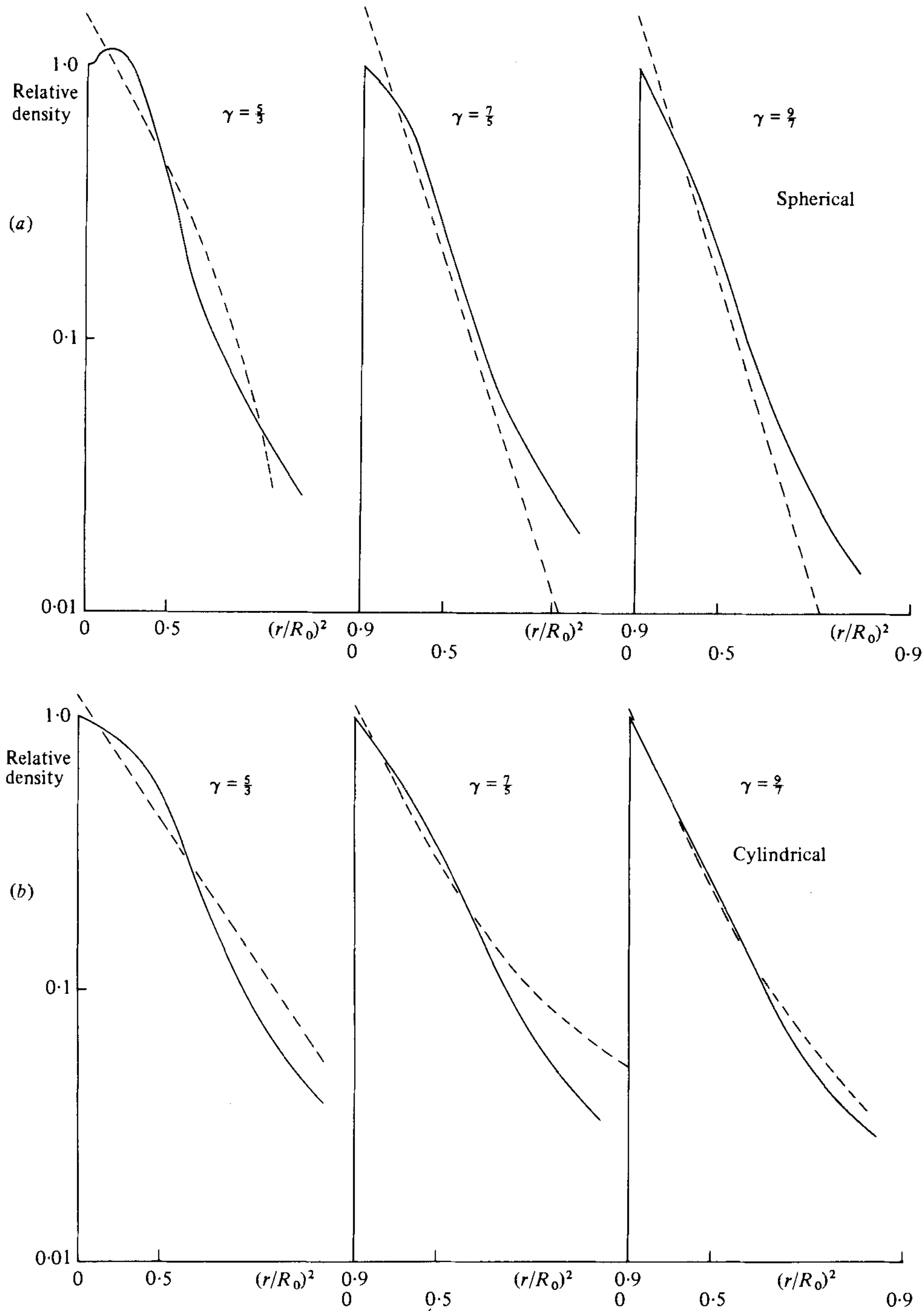


FIGURE 1. Asymptotic density profiles for the adiabatic expansion of polytropic gas spheres (*a*) and cylinders (*b*) of adiabatic index, γ (full line). The self-similar profile at the same time is also plotted (dashed line). The abscissa is taken as $\ln[1-(r/R_0)^2]$ in order to give a clear comparison with the self-similar profile.

Adiabatic similarity model

$$\Psi = \frac{\nu}{2[\nu+2\gamma/(\gamma-1)]}, \quad \begin{cases} \alpha = \dfrac{\Gamma[\frac{1}{2}\nu+\gamma/(\gamma-1)]}{\Gamma(1+\frac{1}{2}\nu)\,\Gamma[\gamma/(\gamma-1)]}, \\ \beta = \dfrac{\Gamma[\gamma/(\gamma-1)]\,\Gamma[\frac{1}{2}\nu+(\gamma+1)/(\gamma-1)]}{\Gamma[(\gamma+1)/(\gamma-1)]\,\Gamma[\nu/2+\gamma/(\gamma-1)]}. \end{cases}$$

γ	Ψ	α	β
Spherical symmetry			
$\frac{5}{3}$	0·16	2·79	1·20
$\frac{7}{5}$	0·156	3·96	1·67
$\frac{9}{7}$	0·154	4·79	1·95
Cylindrical symmetry			
$\frac{5}{3}$	0·146	2·08	1·34
$\frac{7}{5}$	0·135	2·61	1·57
$\frac{9}{7}$	0·129	3·00	1·69

TABLE 1

In figure 2 we compare the behaviour of the average density, $\bar{\rho}$, and the gas edge, r_0,† for the cases (*a*) $\gamma = \frac{5}{3}$ and (*b*) $\gamma = \frac{9}{7}$. It can be seen that for $\gamma = \frac{5}{3}$ the similarity model gives a remarkably good representation of the average parameters of the expansion. For $\gamma = \frac{9}{7}$ a similar conclusion is valid, but the accuracy of the representation is not as high.

Isothermal flows

In the case of isothermal expansions we cannot directly identify the similarity profile with that of the initially uniform mass, since the Gaussian profile is of infinite extent. Two very similar methods of overcoming this difficulty can be proposed.

(*a*) The use of a truncated Gaussian (Haught & Polk 1970) whose edge coincides with the edge of the test mass, r_0, in the asymptotic limit. We define a similarity constant

$$\Gamma_e = \lim_{t\to\infty} R/r_0, \tag{45}$$

where R is the $1/e$ width of the Gaussian.

(*b*) If we can identify an equivalent Lagrangian point in each distribution (for example, whose properties are similar in the asymptotic limit) we may use these to relate the two distributions and define a value of

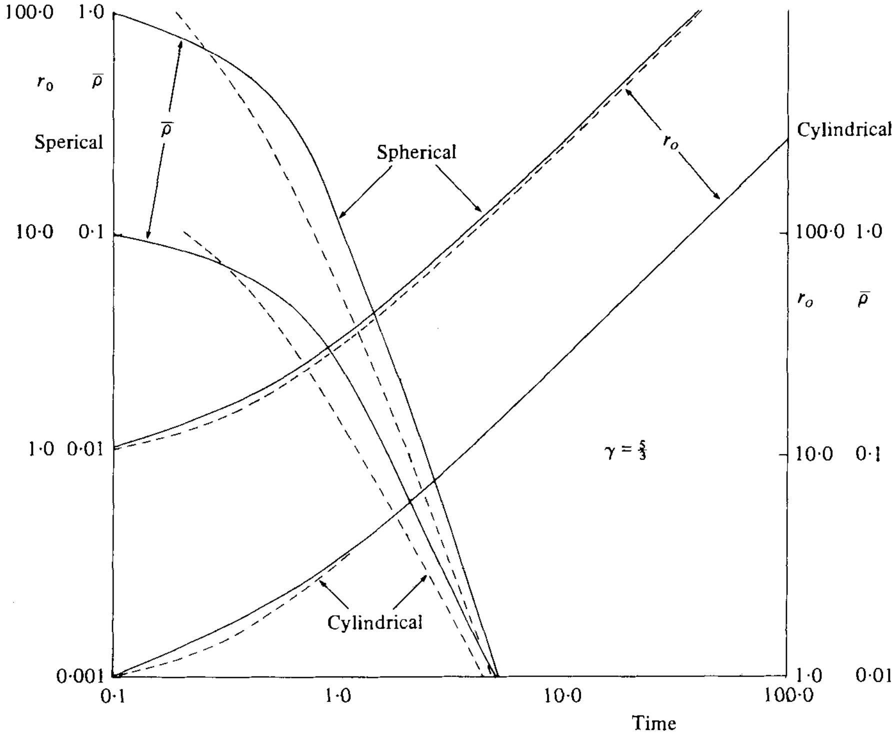
$$\Gamma_d = R(0)/r(0), \tag{46}$$

where $R(0)$ is the $1/e$ width of the Gaussian for which the equivalent points have the same value of r initially.

Cases (*a*) and (*b*) are clearly related, but in case (*b*) the mass distribution remains infinitely extended, whereas it is finite in (*a*); this difference is, however, only significant if $\Gamma \gtrsim 1$.

The asymptotic profiles are shown in figure 3. As for the adiabatic cases the profiles are a reasonable approximation to the Gaussian form. We note, however, that as before the spherical flow for $\gamma = \frac{5}{3}$ shows an off-centre maximum. In each case the

† The initial radius $R(0)$ is taken to be the initial edge of the test distribution $r_0(0)$, i.e. $\Gamma = 1$.



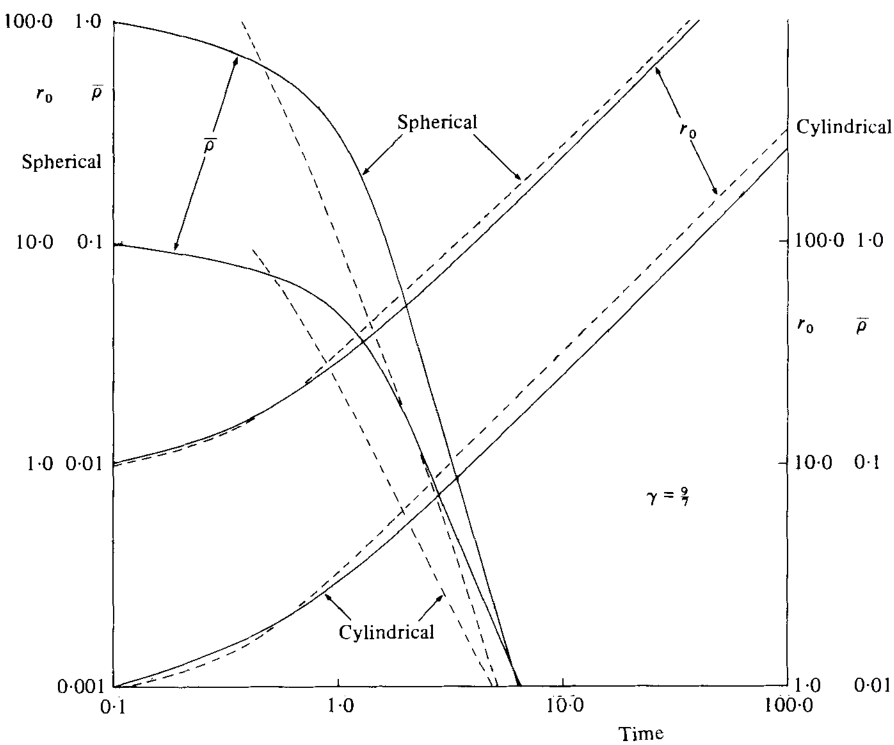


FIGURE 2. Time development of the mean density and edge points during the adiabatic expansion of polytropic gas spheres and cylinders (full line) is compared with the equivalent value calculated for a self-similar expansion (dashed line).

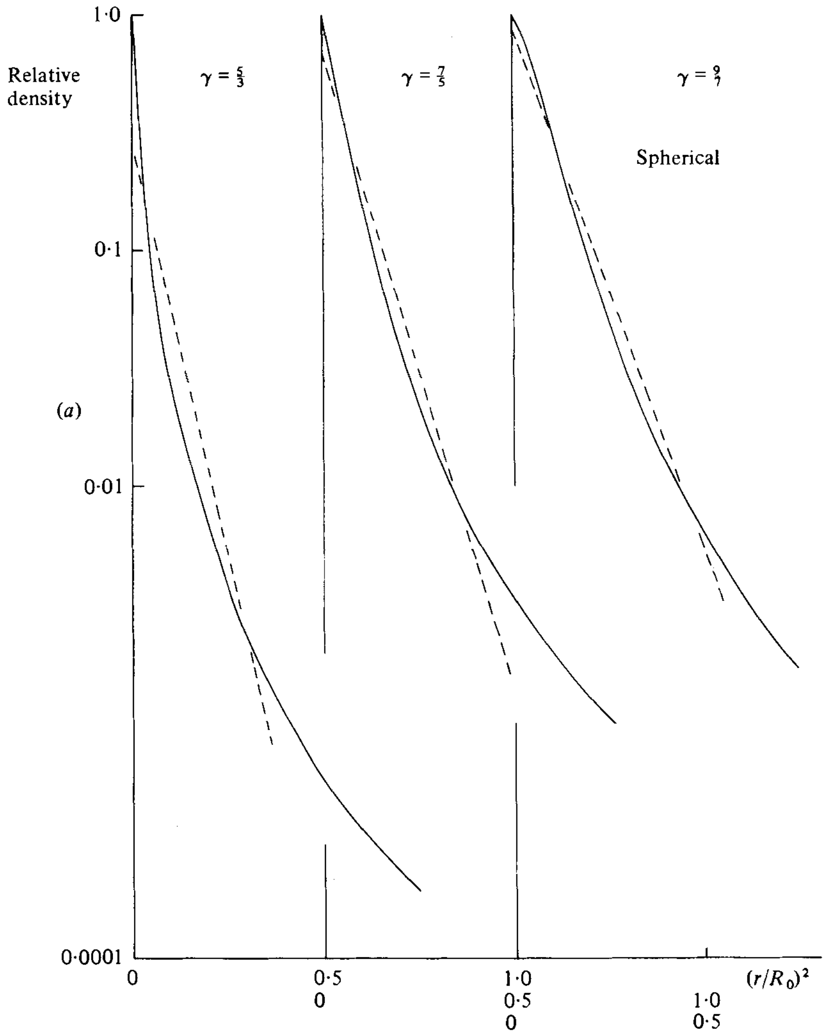


FIGURE 3(*a*). For caption see opposite.

asymptotic profile intersects the test gas one. These intersections represent an equal density point, whose trajectory is identical for both similar and uniform profile. They thus have suitable properties to use as equivalent points. The values of Γ_d so obtained are listed in table 2. Such values have the advantage over those of Γ_e, also given in table 2, that they are not strongly determined by the motion of the outermost cell, which is poorly treated by the Lagrangian code. We note that the asymptotic properties of the self-similar motion are independent of Γ.

In figure 4 we show that development of the characteristic parameters for the uniform profile, compared with those of the similarity model. It can be seen that although the motion of the edge is well described, that of the mean density is relatively inaccurate. This essentially reflects the difference in the asymptotic profiles, figure 3, and cannot be improved. In this context we may remark that the density development

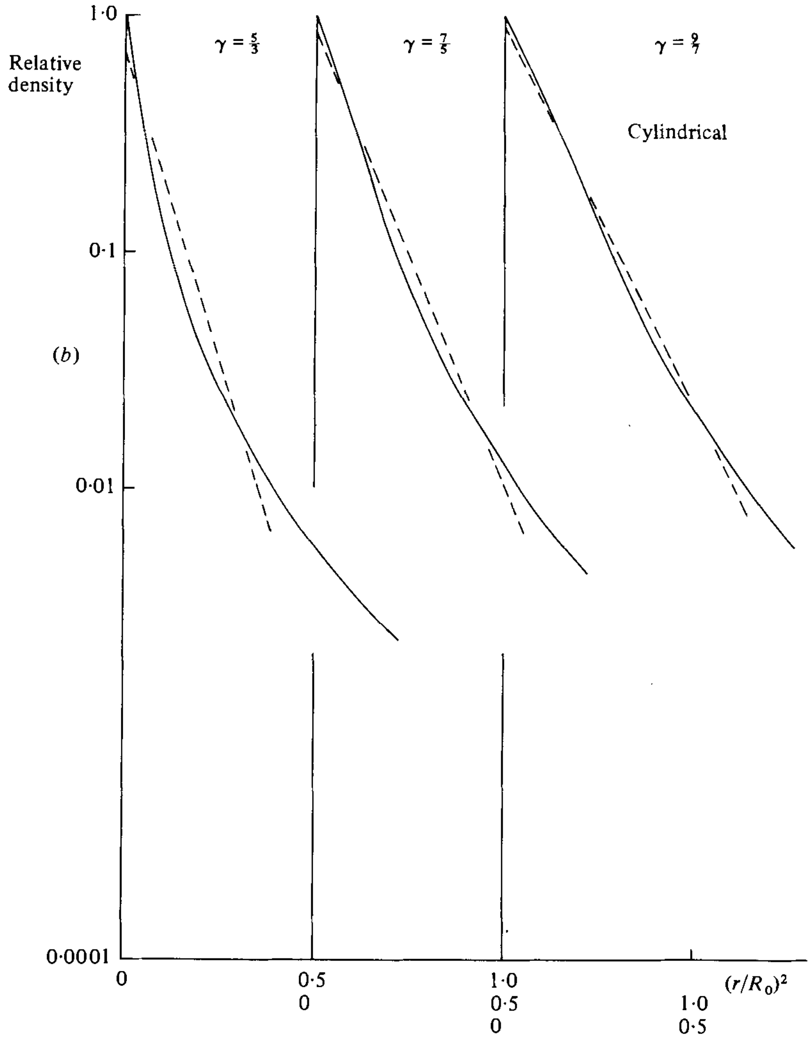


FIGURE 3. Asymptotic density profiles for the isothermal expansion of polytropic gas spheres (*a*) and cylinders (*b*) of adiabatic index, γ (full line). The self-similar profile at the same time is also plotted (dashed line).

is almost independent of Γ for times when $R > r_0(0)$ if $\Gamma \lesssim 0{\cdot}5$, although, of course, the motion of the 'edge' is considerably modified.

Isothermal flow with energy input

The value of the similarity model to reproduce initially uniform gas expansion comes into its own, when we consider flows with a large gradual heat input. As remarked by Nemchinov (1964) we may expect in the asymptotic case of very slow heating of an initially cold uniform mass of gas, that the distribution will closely approximate to the self-similar one. Figure 5 shows that this is indeed the case. We consider the isothermal heating of initially cold (zero specific energy) uniform cylinders and spheres by unit energy/unit mass in a time τ, by a pulse of constant power (E/τ). Comparing figure 5 with figure 3 (which may be considered to be the case of $\tau = 0$) we see clearly

Isothermal similarity model

$$\Psi = \tfrac{1}{4}\Gamma^2\nu, \quad \alpha = \Gamma^{-\nu}\begin{Bmatrix} 2/\sqrt{\pi}, & \nu = 1 \\ 1, & \nu = 2 \\ 4/3\sqrt{\pi}, & \nu = 3 \end{Bmatrix}, \quad \beta = 2^{\frac{1}{2}\nu}$$

γ	Γ_d	Γ_e	Ψ	α	β
Spherical symmetry					
$\frac{5}{3}$	0·456	0·247	0·0457	189·2	3·60
$\frac{7}{5}$	0·444	0·283	0·0614	46·8	2·77
$\frac{9}{7}$	0·458	0·313	0·0735	28·4	2·70
Cylindrical symmetry					
$\frac{5}{3}$	0·514	0·287	0·0412	17·0	2·04
$\frac{7}{5}$	0·490	0·332	0·0550	9·90	1·96
$\frac{9}{7}$	0·496	0·362	0·0654	7·94	1·96

TABLE 2

Heat pulse duration	Γ_d	Γ_e	Ψ	α	β
Spherical symmetry					
0·0	0·456	0·247	0·0457	189·2	3·60
1·0	0·486	0·322	0·0775	28·3	2·57
2·0	0·473	0·354	0·0943	12·9	2.03
5·0	0·453	0·391	0·114	9·80	2·36
10·0	0·524	0·406	0·124	10·1	2·73
100·0	—	0·397	0·116	12·4	2·85
Cylindrical symmetry					
0·0	0·514	0·287	0·0412	17·0	2·04
1·0	0·536	0·359	0·0645	8·00	1·89
2·0	0·511	0·391	0·0763	5·83	1·77
5·0	0·478	0·419	0·0896	4·91	1·84
10·0	0·527	0·435	0·0943	5·00	1·96
100·0	0·489	0·423	0·0862	5·84	2·00

TABLE 3

that as $\tau \to \infty$, the asymptotic distribution becomes more nearly Gaussian. Indeed a good Gaussian form is obtained for $\tau \gtrsim 5$. This is a result of considerable practical value, since the similarity model has been widely used to study laser heating, and verifies Fader's (1968) study of this behaviour. Specifically thus if $\tau \gtrsim 5r_0(0)/(E/M)^{\frac{1}{2}}$ the similarity model provides a good description of the motion. The values of Γ_d and Γ_e thus obtained are given in table 3. We note that values of Γ_e obtained for a spherical distribution are in reasonable agreement with the value of 0·366 given by Fader (1968).

In view of the fact that the case $\gamma = \frac{5}{3}$ showed the most departure from the similarity solution when $\tau = 0$, we may expect that this behaviour, namely the approach to a self-similar form, will occur for smaller values of γ in a corresponding fashion. Test calculations confirm that this is indeed the case.

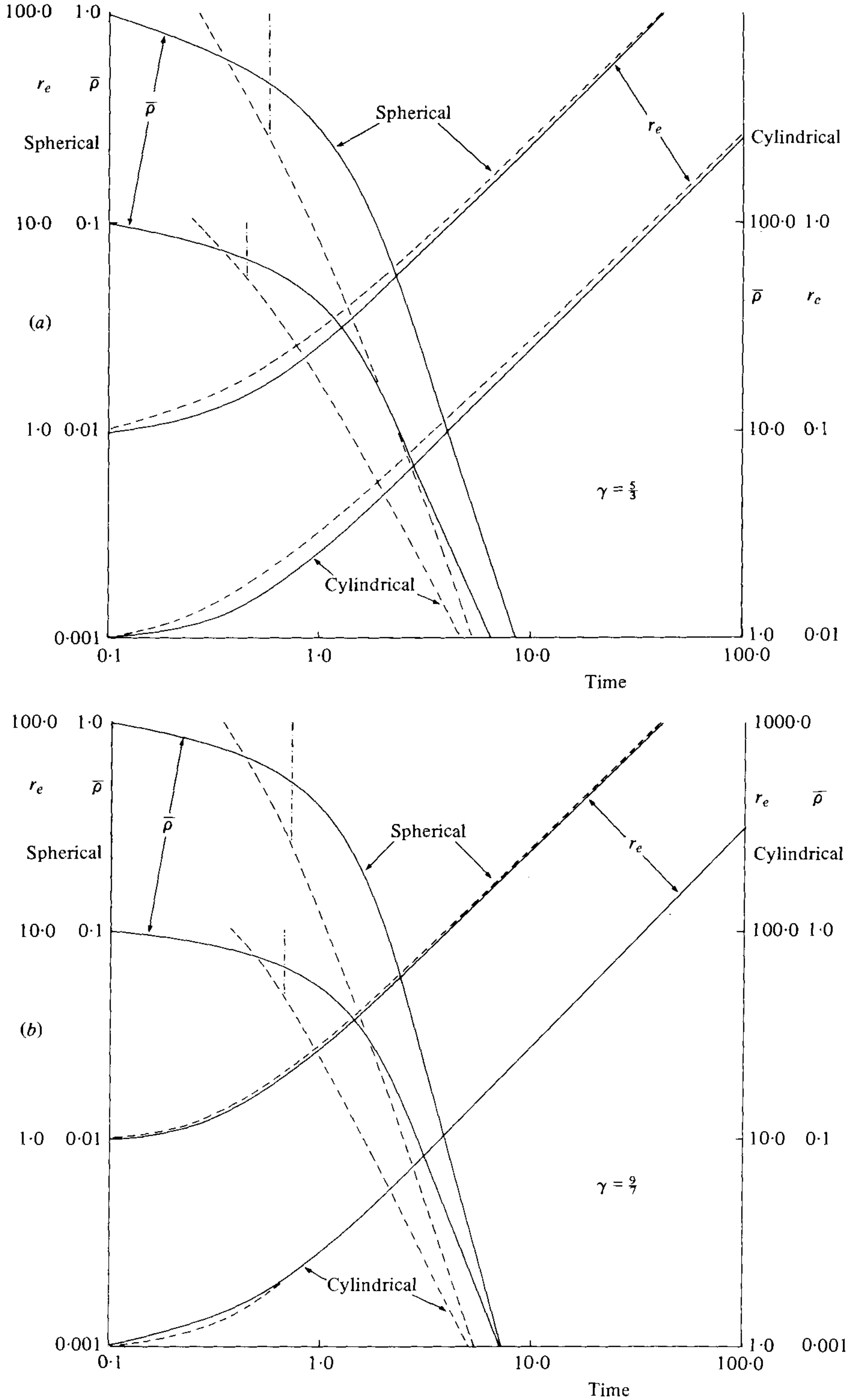


FIGURE 4. Time development of the mean density and 'edge points' during the isothermal expansion of polytropic gas spheres and cylinders (full line) is compared with the equivalent values calculated for a self-similar expansion (dashed line).

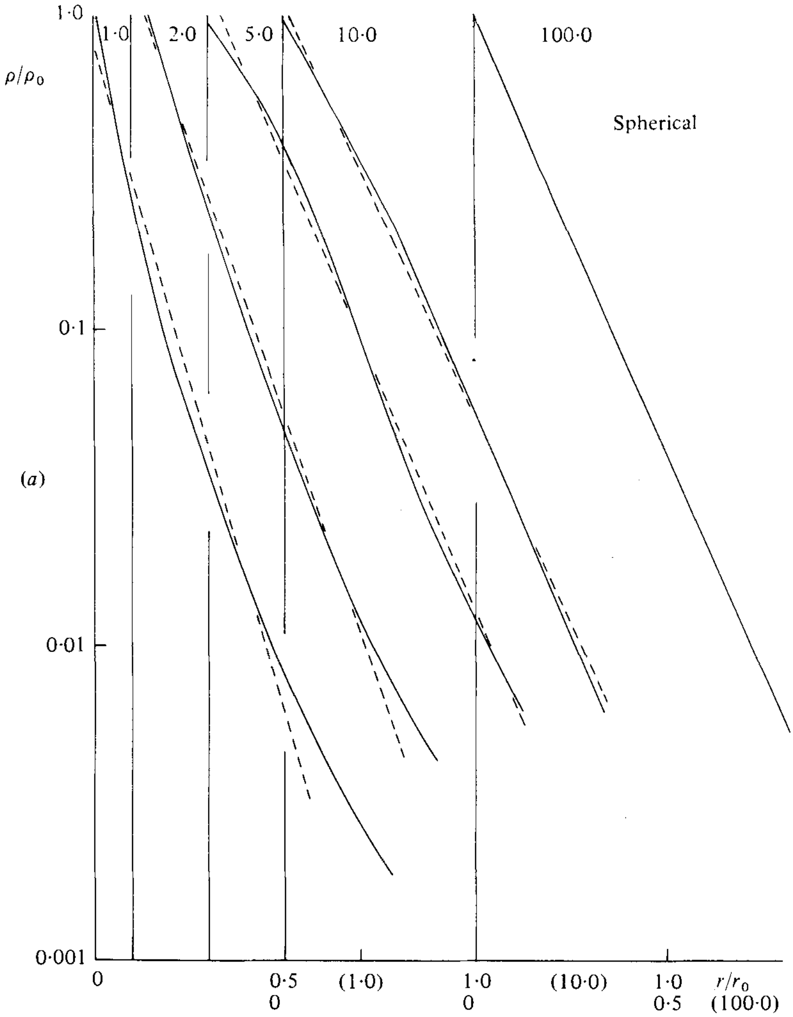


FIGURE 5 (a). For caption see opposite.

6. Discussion

The generalization of the self-similar model of gas expansion to a space of arbitrary dimensions has been shown to be a straightforward extension of the familiar one-dimensional case. The inclusion of additional dimensions of motion in general, however, leads to a density distribution function which is a function of the ν Lagrangian co-ordinates of the gas. This function is determined by the thermodynamic constraints on the motion, for example, by the deposition of heat from point-to-point in the fluid. In most cases of interest the constraints specified are isotropic in the Lagrangian co-ordinates, in which case the density distribution is ellipsoidal, being a function of the Lagrangian radius vector, ζ. We note that the initially ellipsoidal distribution remains so throughout the expansion, although the eccentricities change in time. We have shown that this motion is peculiar to self-similar motions with a linear velocity gradient, and have therefore called such flows ellipsoidal.

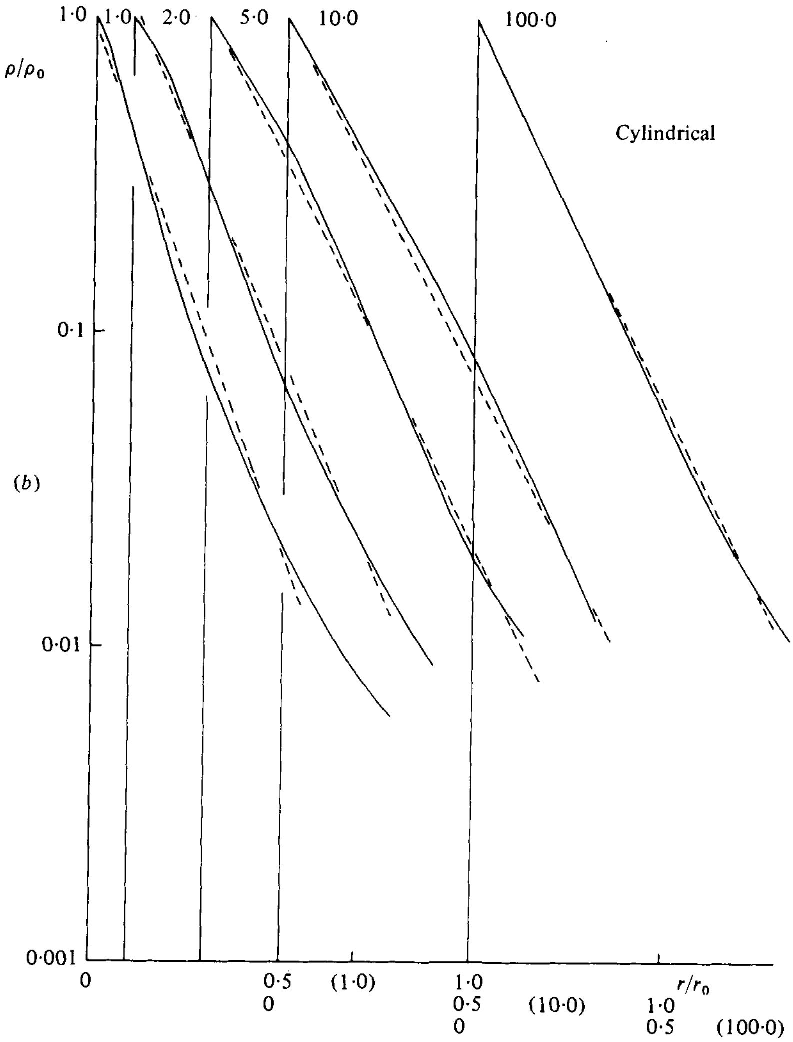


FIGURE 5. Asymptotic density profiles for the isothermal expansion of heated spheres (*a*) and cylinders (*b*) of adiabatic index $\frac{5}{3}$. The curves are characterized by the heating time. The dashed line shows the equivalent self-similar profile at the same time.

It has been shown that the expansion into vacuum of any body leads asymptotically to a self-similar form in the dimensions of the space of the flow. This result has been used to suggest that an approximate representation of uniform gas masses may be obtained by the use of the self-similar model. The relative mathematical simplicity of the self-similar equations make this an attractive approach for all but the simplest planar problems, for which exact analytic solutions are available. As a result this representation has been widely used to study cylindrical and spherical expansions.

The validity of this hypothesis applied to the important cases of one dimensional spherical and cylindrical flow into vacuum has been examined by comparing self-similar flows with those calculated by a numerical fluid code.

Parameters for the modelling of such expansions based on either the asymptotic form, or the equivalent self-similar solution are given. From a practical point-of-view we have verified, for the restricted case of an isothermal polytropic gas of $\gamma = \frac{5}{3}$, Nemchinov's (1964) assertion that if the heating time is long, the motion is self-similar throughout the times of interest, and that the asymptotic profile is the equivalent self-similar profile. This result is also in agreement with the work of Fader (1968) who considered the specific problem of laser heated pellets. In order to match the test expansion to a model isothermal flow it is necessary to define a matching parameter, Γ. Two ways of doing this have been discussed. Fortunately, however, the properties of the expansion are only weakly dependent on Γ, if condition (42) is included, since the value of Γ is only poorly known. This is due to the fact that the motion of the fluid edge is poorly treated by the Lagrangian code, and further that the self-similar description is poor there. An alternative prescription for Γ was suggested to avoid these difficulties, but is also not entirely satisfactory. The value of Γ obtained for slow heating agrees reasonably well with Fader's (1968) result of 0·366.

The extension of these representations to two or more dimensions is not discussed in this paper. Apart from the numerical problems of modelling the test flows, there are problems with the asymptotic representation. These arise since the asymptotic forms are not necessarily ellipsoidal, even if the initial distribution was, by virtue of the axiom discussed earlier. The equivalent self-similar distributions on the other hand are. However, this is probably not very important since the equivalence is only supposed to exist in a coarse averaged sense, and is probably not sensitive to the exact form of the distribution: a result already observed in the one-dimensional case through the insensitivity of the parameters to Γ. Alternatively the problem may be approached through the use of the asymptotic profile as the self-similar distribution, although this may not be a very flexible technique unless the appropriate averages for the problem considered are known.

An analysis of the stability of these self-similar flows with spherical symmetry has recently been published by Book (1979), in which it is concluded that the flow is unstable only if the specific entropy decreases outwards. In general both isentropic and isothermal flows of the type considered here are therefore stable.

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